The Potts model and the Beraha numbers

1987 J. Phys. A: Math. Gen. 20 L399

(http://iopscience.iop.org/0305-4470/20/6/011)

J. Phys. A: Math. Gen. **20** (1987) L399-L403. Printed in the UK

LETTER TO THE EDITOR

The Potts model and the Beraha numbers

P P Martin

Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK

Received 29 October 1986

Abstract. We show explicitly that the representation of a finite-dimensional algebra associated with the largest eigenvalues of the four-site transfer matrix for the q-state Potts model is reducible at the first irrational Beraha q value. We show how this fits into a pattern of reducibility at the Beraha values so that the asymptotic dimensionality of representations for large lattice width n is q^n. For other q values the asymptotic dimensionality is 4^n $(q>4)$ or K^n with $K>q$ $(0<q<4)$.

Several authors have recently observed (directly or indirectly) that the Potts models associated with the leading discrete infinity of conformal field theories in two dimensions are those with the Beraha q values (Friedan *et al* 1984, Dotsenko and Fateev 1984, Kuniba *et al* 1986). The latter authors, in particular, noted an apparent connection with the properties of a finite-dimensional von Neumann algebra (see also Jones 1983, 1985). This algebra is associated with the transfer matrix for the Potts model (Baxter 1982). In a recent letter (Martin 1986a and later corrigendum) we showed how to write down the representation of this algebra for the largest eigenvalue of the transfer matrix. We now show how this approach can indeed pick out the Beraha values, since the representation is reducible at these values. This leads to a distinctive dimensionality for the resultant irreducible representation (it is easy to see, for instance from Martin (1986b), that this dimensionality has a strong bearing on the form of the transfer matrix eigenvalues at all temperatures).

A representation for the operators $\{U_i, i=1,\ldots,2n-1\}$ satisfying

$$\begin{aligned} &U_i^2=q^{1/2}U_i \\ &U_iU_{i\pm1}U_i=U_i \\ &U_iU_j=U_jU_i \qquad |i-j|\geqslant 2 \end{aligned} \tag{1}$$

is given in Martin (1986a). The representation dimension C_n is given by

$$\begin{aligned} &C_1=1 \\ &C_n=\frac{4n-2}{n+1}C_{n-1} \end{aligned} \tag{2}$$

and the representation has the property

$$R\equiv\prod_{i\ \mathrm{odd}} U_i\neq 0. \tag{3}$$

0305-4470/87/060399+05$02.50

When $q=4$ the operators

$$J_i = 1 - U_i \tag{4}$$

generate the permutation group S_{2n} (see also Temperley 1986). By considering all possible partitions of Young tableaux of two equal rows $T^{[n\,n]}$ into $T^{[n-m\,n-m]}$ and $T^{[m\,m]}$ it is then possible to see that the representation is irreducible. Since the representation is, up to overall constants, polynomial in q it then remains irreducible for finite n and general q, except possibly at a finite set of q values.

In Martin (1986a) we showed that at $n=3,5$ this set includes some of the points ($q=1,2,3$) where Temperley's (1986) generalised Young tableaux construction breaks down. These are, in full,

$$q = 4\cos^2\left(\frac{\pi m}{r}\right) \qquad \text{for any } m = 1, \ldots, r-1;\ r = 2, \ldots, n. \tag{5}$$

We conjectured that the representation might reduce at these points for general n, although the mechanism was unclear since all the other q values involved are non-integer.

We can clarify this issue by reference to the $n=4$ case. Here there exists a basis such that

$$U_1 = \text{diagonal}(1,1,1,0,0,1,0,1,0,0,0,0,0,0)q^{1/2}$$

$$U_3 = \text{diagonal}(1,1,0,0,1,0,1,0,0,0,1,0,0,0)q^{1/2}$$

$$U_5 = \text{diagonal}(1,0,0,1,1,1,0,0,0,1,0,0,0,0)q^{1/2}$$

$$U_7 = \text{diagonal}(1,0,1,1,0,0,1,0,1,0,0,0,0,0)q^{1/2}$$

$$U_2 = q^{-1/2}\begin{pmatrix}
1 & & & 1 & & & & & & & & & & \\
 & 1 & & & & & & & & & & 1 & & \\
 & & 1 & & & & 1 & & 1 & & & & & \\
q-1 & & & q-1 & & & & & & & & & & \\
 & & & & 1 & 1 & & & & 1 & & & & \\
 & & & & 1 & 1 & & & & 1 & & & & \\
 & & 1 & & & & 1 & & 1 & & & & & \\
 & & & & & & & 1 & & & 1 & q-3 & q-2 & 1 \\
 & & q-2 & & & & q-2 & & q-2 & & & & & \\
 & & & & q-2 & q-2 & & & & q-2 & & & & \\
 & & & & & & & 1 & & & 1 & q-3 & q-2 & 1 \\
 & q-1 & & & & & & & & & & q-1 & & \\
 & 1 & & & & & & 1 & & & 1 & q-2 & q-2 & 1 \\
 & a & & & & & & & & & & a & & 0
\end{pmatrix}$$

$$U_4 = q^{-1/2}\begin{pmatrix}
1 & & 1 & & & & & & & & & & & \\
 & 1 & & & & 1 & & 1 & & & & & & \\
q-1 & & q-1 & & & & & & & & & & & \\
 & & & 1 & & & 1 & & 1 & & & & & \\
 & & & & 1 & & & & & & & & 1 & \\
 & 1 & & & & 1 & & 1 & & & & & & \\
 & & & 1 & & & 1 & & 1 & & & & & \\
 & q-2 & & & & q-2 & & q-2 & & & & & & \\
 & & & q-2 & & & q-2 & & q-2 & & & & & \\
 & & & & & & & & & 1 & 1 & q-2 & q-3 & 1 \\
 & & & & & & & & & 1 & 1 & q-2 & q-3 & 1 \\
 & & & & 1 & & & & & 1 & 1 & q-2 & q-2 & 1 \\
 & & & & q-1 & & & & & & & & q-1 & \\
 & & & & a & & & & & & & & a & 0
\end{pmatrix}$$

$$U_6 = q^{-1/2}\begin{pmatrix}
1 & 1 & & & & & & & & & & & & \\
q-1 & q-1 & & & & & & & & & & & & \\
 & & 1 & & & 1 & & 1 & & & & & & \\
 & & & 1 & & & & & & & & 1 & & \\
 & & & & 1 & & 1 & & & & 1 & & & \\
 & & 1 & & & 1 & & 1 & & & & & & \\
 & & & & 1 & & 1 & & & & 1 & & & \\
 & & q-2 & & & q-2 & & q-2 & & & & & & \\
 & & & & & & & & 1 & 1 & & q-3 & q-2 & 1 \\
 & & & & & & & & 1 & 1 & & q-3 & q-2 & 1 \\
 & & & & q-2 & & q-2 & & & & q-2 & & & \\
 & & & q-1 & & & & & & & & q-1 & & \\
 & & & 1 & & & & & 1 & 1 & & q-2 & q-2 & 1 \\
 & & & a & & & & & & & & a & & 0
\end{pmatrix}$$

with $a = 1-(q-1)(q-2)$. (6)

We see that the representation is reducible when $q = 1, 2$ and when

$$(q-1)(q-2) = 1$$

i.e. when

$$\prod_{k=1}^{[n/2]}\left[q - 4\cos^2\left(\frac{\pi k}{n+1}\right)\right] = 0 \tag{7}$$

where $[p]$ is the integer part of p (the roots are $q = (3\pm\sqrt{5})/2$). The generalisation to larger n is indicated in the form of (7), which is always a polynomial in q with integer coefficients.

We denote the dimension of the irreducible representation at a given n for the set

$$\left\{q = 4\cos^2\left(\frac{\pi m}{r}\right),\ m = 1, \ldots, [(r-1)/2]\right\} \tag{8}$$

by ${}^{r}C_n$. Then using results from Martin (1986a) and Blote and Nightingale (1982) it can be seen that the function

$$B_r(x) \equiv 1+\sum_{n=1}^{\infty} {}^{r}C_n x^n$$

$$= \prod_{j=1}^{[(r-2)/2]}\left[1-4\cos^2\left(\frac{\pi j}{r-1}\right)x\right]\left\{\prod_{l=1}^{[(r-1)/2]}\left[1-4\cos^2\left(\frac{\pi l}{r}\right)x\right]\right\}^{-1} \tag{9}$$

for $r=3,4,6$ and ∞ (an integral familiar from the solution of the Ising model is useful in the latter case—see McCoy and Wu (1973)). For all integer $r\geqslant 3$ this function has the property that ${}^{r}C_n$ deviates from ${}^{\infty}C_n$ at $r=n-1$ (cf equation (5)), and, for example, ${}^{5}C_4=13$ as required. By choosing an appropriate basis in the critical $(r-1)$-state Andrews-Baxter-Forrester model (Andrews *et al* 1984) it is easy to extract representations with $R\neq 0$ which have the dimensions given in (9). These are therefore at least an upper bound on the true irreducible dimensions (see Martin (1986a) and note that the operators U are temperature independent). Furthermore the asymptotic behaviour of ${}^{r}C_n$ for large n is then

$${}^{r}C_n \sim \left[4\cos^2\left(\frac{\pi}{r}\right)\right]^n. \tag{10}$$

Equation (9) gives the only suitable function with monotonic convergence to this behaviour.

It can then be seen (again using Martin (1986a) and Andrews *et al* (1984)), with

$$e_i = q^{-1/2}U_i \tag{11}$$

and ${}^{r}e_i$ denoting the above (irreducible) representations of e_i for $q=4\cos^2(\pi/r)$, that we have

$$\mathrm{Tr}\, {}^{r}e_i = \frac{{}^{r}C_{n-1}}{{}^{r}C_n} \underset{n\to\infty}{\to} \frac{1}{q} \tag{12}$$

($\mathrm{Tr}\, 1 \equiv 1$). Thus in the thermodynamic limit these representations realise Jones' (1983) trace condition for integer $r\geqslant 3$.

We have briefly indicated how the Beraha Potts models are picked out in the statistical mechanical context, and how this can provide a realisation for Jones' trace condition. It also provides an intriguing framework for examining the connection between the von Neumann algebra and the Virasaro algebra associated with conformal symmetry in two dimensions (cf Kuniba *et al* 1986).

I would like to thank David Holling for useful discussions.

References

Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
Blöte H W J and Nightingale M P 1982 *Physica* **112A** 405
Dotsenko V L S and Fateev V A 1984 *Nordita preprint* 84/8
Friedan D, Qiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
Jones V F R 1983 *Invent. Math.* **72** 1
—— 1985 *Bull. Am. Math. Soc.* **12** 103

Kuniba A, Akutsu Y and Wadati M 1986 *Virasoro Algebra, Von Neumann Algebra and Critical 8-Vertex SOS Models. Kanagawa preprint*

Martin P P 1986a *J. Phys. A: Math. Gen.* **19** L1117

—— 1986b *J. Phys. A: Math. Gen.* **19** 3267

McCoy B M and Wu T T 1973 *The Two Dimensional Ising Model* (Cambridge, MA: Harvard University Press)

Temperley H N V 1986 *Potts Models and Related Problems. Preprint*